WESLEY COLLEGE

By daring & by doing

YEAR 12 MATHEMATICS SPECIALIST
SEMESTER ONE 2017
QUESTIONS OF REVIEW 4:
Vector Calculus, Equations and Applications

Name: ______________________

Wednesday 31st May   Time: ~~35~~ 40 minutes   Mark   /35

Calculator allowed.

1. [7 marks – 2, 1, 2 and 2]

A particle is moving through a 3 dimensional space with velocity given by the vector $\overrightarrow{v(t)} = 6t\,i + (8t-5)\,j + 3k$

a) Determine $\overrightarrow{r(t)}$ given that the particle started at $(0, -3, 0)$

$\vec{r}(t) = 3t^2 i + (4t^2 - 5t - 3)j + 3tk$

b) Write down an expression for the acceleration vector $\overrightarrow{a(t)}$

$\vec{a}(t) = 6i + 8j$

c) Decide if, and when, the acceleration is perpendicular to the direction of motion

$\begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 6t \\ 8t-5 \\ 3 \end{bmatrix} = 0 \Rightarrow 36t + 64t - 40 = 0$

$t = 0.4$

$\therefore$ are $\perp$, at $t = 0.4$

d) Calculate the distance travelled in the first 5 seconds of motion.

$\text{Distance} = \int_0^5 \text{norm}\left(\vec{v}(t)\right) dt$

$= \int_0^5 \text{norm}\,[6x \quad 8x-5 \quad 3]\,dx$ or $\int_0^5 \sqrt{36t^2 + (8t-5)^2 + 9}\,dt$

$= 110.9$ units

2. [10 marks – 4, 1, 1, 2 and 2]

A child's model train is moving on a track with position given by

$$\overrightarrow{r_C} = 2\sin\left(\frac{\pi}{6}t\right)i + \left(2 - 2\cos\left(\frac{\pi}{6}t\right)\right)j$$

a) Describe its motion in terms of:

shape of the track — circular (centre $(0,2)$, $r = 2$)

direction of travel — anticlockwise from $(0,0)$

period of motion — 12 units

{Hints: Zoom initialize, set $t_{\max} \approx 20$}

b) Determine a Cartesian equation to represent the shape of the track.

$$x^2 + (y-2)^2 = 4$$

c) Specify $\overrightarrow{v(t)}$, the velocity vector

$$\vec{v}(t) = \frac{\pi}{3}\cos\frac{\pi t}{6}\, i + \frac{\pi}{3}\sin\frac{\pi t}{6}\, j$$

d) How far does the train travel in 24 seconds?

$$2 \text{ circuits} = 2 \times 2 \times \pi \times 2$$

$$= 8\pi \text{ units}$$

e) Calculate the maximum and minimum values of the train's speed.

$$|\vec{v}(t)| = \frac{\pi}{3} \quad (\text{constant})$$

$$\therefore \text{min} = \text{max} = \frac{\pi}{3} \text{ units}$$

3. [9 marks – 3, 3 and 3]

(a) Complete the indicated elementary row operations and bring the augmented matrix to echelon form:

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 2 & k & 3 & k-1 \\ 3 & 2 & k+3 & 1 \end{array}\right] \rightarrow \begin{bmatrix} R_1 \\ R_2 - 2R_1 \\ R_3 - 3R_1 \end{bmatrix} \rightarrow \begin{bmatrix} R_1 \\ R_2 \\ kR_3 - 2R_2 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & k & -1 & k-1 \\ 0 & 2 & k-3 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & k & -1 & k-1 \\ 0 & 0 & k^2-3k+2 & -k+2 \end{array}\right]$$

(b) Use this echelon matrix to solve $\begin{cases} x+2z=0 \\ 2x+4y+3z=3 \\ 3x+2y+7z=1 \end{cases}$

$k = 4$

$\therefore\ 6z = -2$

$\therefore\ z = -\frac{1}{3}$

$4y + \frac{1}{3} = 3$

$\therefore\ y = \frac{2}{3}$

& $x = \frac{2}{3}$

(c) For which values of k will $\begin{cases} x+2z=0 \\ 2x+ky+3z=k-1 \\ 3x+2y+(k+3)z=1 \end{cases}$ have:

(i) no solutions

$k^2 - 3k + 2 = 0$ and $-k+2 \neq 0$

$(k-2)(k-1) = 0$

$\therefore\ k = 1$ only.

(ii) a unique solution

All k except $k=1$ & $k=2$

i.e. $k \in \mathbb{R}$, $k \neq 1$, $k \neq 2$

4. [9 marks – 3, 2, 1 and 3]

a) Use elementary row operations to determine the number of solutions to the system of equations represented by the augmented matrix $\left[\begin{array}{ccc|c} 0 & -2 & -1 & -6 \\ 2 & 0 & -3 & 14 \\ 1 & 3 & 0 & 16 \end{array}\right]$

$$\begin{array}{r} \\ 2R_3 - R_2 \\ 3R_1 + \text{new } R_2 \end{array} \left[\begin{array}{ccc|c} 0 & -2 & -1 & -6 \\ 0 & 6 & 6 & 18 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\Rightarrow$ infinite number of solutions.

Given that $\vec{a} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and $\vec{a} \times \vec{v} = \begin{bmatrix} -6 \\ 14 \\ 16 \end{bmatrix}$

b) Explain why the system $\begin{cases} -2y - z = -6 \\ 2x - 3z = 14 \\ x + 3y = 16 \end{cases}$ represents this situation

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \times \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -z - 2y \\ 2x - 3z \\ 3y + x \end{bmatrix} = \begin{bmatrix} -6 \\ 14 \\ 16 \end{bmatrix} \Rightarrow \begin{array}{l} -2y - z = -6 \\ 2x - 3z = 14 \\ x + 3y = 16 \end{array}$$

c) Write down an equation to represent $\vec{a} \bullet \vec{v} = -10$

$3x - y + 2z = -10$

d) Determine $\vec{v}$

Solve on ClassPad $\quad x = 1 \quad y = 5 \quad z = -4$

$\therefore \vec{v} = \begin{bmatrix} 1 \\ 5 \\ -4 \end{bmatrix}$